and

$$\beta^2 u_{i,j-1}^{k+1} - 2(1+\beta^2)u_{i,j}^{k+1} + \beta^2 u_{i,j+1}^{k+1} = -\left(u_{i+1,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+1}\right) \tag{4.1.14b}$$

Here (4.1.14a) and (4.1.14b) are solved implicitly in the x-direction and y-direction, respectively. The relaxation parameter ω may be introduced to accelerate the convergence,

$$\omega u_{i-1,j}^{k+\frac{1}{2}} - 2(1+\beta^2)u_{i,j}^{k+\frac{1}{2}} + \omega u_{i+1,j}^{k+\frac{1}{2}} = -(1-\omega)[2(1+\beta^2)]u_{i,j}^{k} - \omega\beta^2\left(u_{i,j+1}^{k} + u_{i,j-1}^{k+\frac{1}{2}}\right) \tag{4.1.15a}$$

and

$$\omega\beta^2 u_{i,j-1}^{k+1} - 2(1+\beta^2)u_{i,j}^{k+1} + \omega\beta^2 u_{i,j+1}^{k+1} = -(1-\omega)[2(1+\beta^2)]u_{i,j}^{k+\frac{1}{2}} - \omega\left(u_{i+1,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+1}\right) \tag{4.1.15b}$$

with the optimum ω being determined experimentally as appropriate for different physical problems.

4.1.3 DIRECT METHOD WITH GAUSSIAN ELIMINATION

Consider the simultaneous equations resulting from the finite difference approximation of (4.1.2) in the form

$$\begin{aligned} k_{11}u_1 + k_{12}u_2 + \cdots &= g_1 \\ k_{21}u_1 + k_{22}u_2 + \cdots &= g_2 \\ \vdots \\ k_{n1}u_n \cdot \qquad \cdots &= g_n \end{aligned} \tag{4.1.16}$$

Here, our objective is to transform the system into an upper triangular array. To this end, we choose the first row as the "pivot" equation and eliminate the u_1 term from each equation below it. To eliminate u_1 from the second equation, we multiply the first equation by k_{21}/k_{11} and subtract it from the second equation. We continue similarly until u_1 is eliminated from all equations. We then eliminate $u_2, u_3, \ldots$ in the same manner until we achieve the upper triangular form,

$$\begin{bmatrix} k_{11} & k_{12} & \cdot & \cdot \\ & k'_{22} & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & k'_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g'_2 \\ \cdot \\ g'_n \end{bmatrix} \tag{4.1.17}$$

It is seen that backsubstitution will determine all unknowns.

An example for the solution of a typical elliptical equation is shown in Section 4.7.1.

4.2 PARABOLIC EQUATIONS

The governing equations for some problems in fluid dynamics, such as unsteady heat conduction or boundary layer flows, are parabolic. The finite difference representation

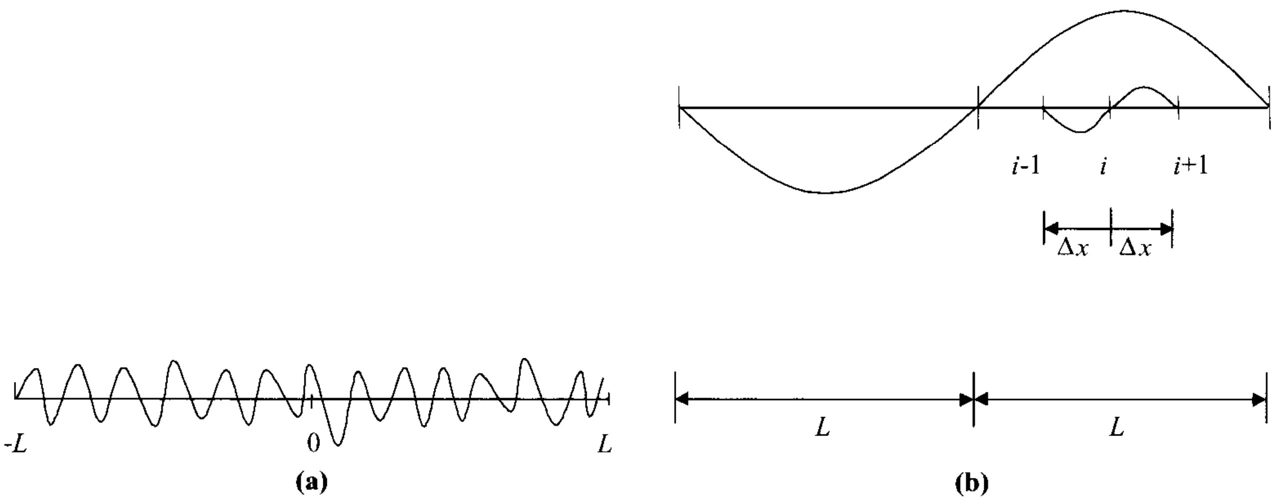

Figure 4.2.1 Fourier representation of the error on interval $(-L, L)$. **(a)** Error distribution. **(b)** Maximum and minimum wavelength.

of these equations may be represented in either explicit or implicit schemes, as illustrated below.

4.2.1 EXPLICIT SCHEMES AND VON NEUMANN STABILITY ANALYSIS

Forward-Time/Central-Space (FTCS) Method

A typical parabolic equation is the unsteady diffusion problem characterized by

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \tag{4.2.1}$$

An explicit finite difference equation scheme for (4.2.1) may be written in the forward difference in time and central difference in space (FTCS) as (see Figure 4.2.1a)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{\Delta x^2} + O(\Delta t, \Delta x^2) \tag{4.2.2a}$$

or

$$u_i^{n+1} = u_i^n + d\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \tag{4.2.2b}$$

where d is the diffusion number

$$d = \frac{\alpha \Delta t}{\Delta x^2} \tag{4.2.3}$$

By definition, (4.2.2) is explicit because u_i^{n+1} at time step $n+1$ can be solved *explicitly* in terms of the known quantities at the previous time step n, thus called an *explicit scheme*.

In order to determine the stability of the solution of finite difference equations, it is convenient to expand the difference equation in a Fourier series. Decay or growth of an amplification factor indicates whether or not the numerical algorithm is stable. This is known as the von Neumann stability analysis [Ortega and Rheinbolt, 1970]. Assuming that at any time step n, the computed solution u_i^n is the sum of the exact solution $\bar{u}_i^n$ and error ε_i^n

$$u_i^n = \bar{u}_i^n + \varepsilon_i^n \tag{4.2.4}$$

and substituting (4.2.4) into (4.2.2a), we obtain

$$\frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\Delta t} + \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}(\bar{u}_{i+1}^n - 2\bar{u}_i^n + \bar{u}_{i-1}^n) + \frac{\alpha}{(\Delta x)^2}(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n) \tag{4.2.5}$$

or

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n) \tag{4.2.6}$$

Writing (4.2.4) – (4.2.6) for the entire domain leads to

$$\mathbf{U}^n = \bar{\mathbf{U}}^n + \boldsymbol{\varepsilon}^n \tag{4.2.7}$$

with

$$\boldsymbol{\varepsilon}^n = \begin{bmatrix} \cdot \\ \varepsilon_{i-1}^n \\ \varepsilon_i^n \\ \varepsilon_{i+1}^n \\ \cdot \end{bmatrix} \tag{4.2.8}$$

$$\bar{\mathbf{U}}^{n+1} + \boldsymbol{\varepsilon}^{n+1} = \mathbf{C}(\bar{\mathbf{U}}^n + \boldsymbol{\varepsilon}^n) \tag{4.2.9}$$

$$\boldsymbol{\varepsilon}^{n+1} = \mathbf{C}\boldsymbol{\varepsilon}^n \tag{4.2.10}$$

with

$$C = 1 + d(E - 2 + E^{-1}) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ d & (1-2d) & d & 0 & 0 \\ \cdot & d & (1-2d) & d & 0 \\ \cdot & 0 & d & (1-2d) & d \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{4.2.11}$$

If the boundary conditions are considered as periodic, the error ε^n can be decomposed into a Fourier series in space at each time level n. The fundamental frequency in a one-dimensional domain between $-L$ and L (Figure 4.2.1) corresponds to the maximum wave length of $\lambda_{\max} = 2L$. The wave number $k = 2\pi/\lambda$ becomes minimum as $k_{\min} = \pi/L$, whereas the maximum wave number $k_{\max}$ is associated with the shortest wavelength λ on a mesh with spacing Δx corresponding to $\lambda_{\min} = 2\Delta x$, leading to $k_{\max} = \pi/\Delta x$. Thus, the harmonics on a finite mesh are

$$k_j = jk_{\min} = j\pi/L = j\pi/(N\Delta x), \quad j = 0, 1, \ldots N \tag{4.2.12}$$

with $\Delta x = L/N$. The highest value of j is equal to the number of mesh intervals N. Any finite mesh function, such as ε_i^n or the full solution u_i^n, can be decomposed into a Fourier series

$$\varepsilon_i^n = \sum_{j=-N}^{N} \bar{\varepsilon}_j^n \, e^{Ik_j(i\Delta x)} = \sum_{j=-N}^{N} \bar{\varepsilon}_j^n \, e^{Iji\pi/N} \tag{4.2.13}$$

with $I = \sqrt{-1}$, $\bar{\varepsilon}_j^n$ being the amplitude of the j^{th} harmonic, and the spatial phase angle ϕ is given as

$$\phi = k_j \Delta x = j\pi/N \tag{4.2.14}$$

with $\phi = \pi$ corresponding to the highest frequency resolvable on the mesh, namely the frequency of the wavelength $2\Delta x$. Thus

$$\varepsilon_i^n = \sum_{j=-N}^{N} \bar{\varepsilon}_j^n \, e^{Ii\phi} \tag{4.2.15}$$

Substituting (4.2.15) into (4.2.6) yields

$$\frac{\bar{\varepsilon}^{n+1} - \bar{\varepsilon}^n}{\Delta t} e^{Ii\phi} = \frac{\alpha}{\Delta x^2}\left(\bar{\varepsilon}^n e^{I(i+1)\phi} - 2\bar{\varepsilon}^n e^{Ii\phi} + \bar{\varepsilon}^n e^{I(i-1)\phi}\right)$$

or

$$\bar{\varepsilon}^{n+1} - \bar{\varepsilon}^n - d\bar{\varepsilon}^n(e^{I\phi} - 2 + e^{-I\phi}) = 0 \tag{4.2.16}$$

The computational scheme is said to be stable if the amplitude of any error harmonic $\bar{\varepsilon}^n$ does not grow in time, that is, if the following ratio holds:

$$|g| = \left|\frac{\bar{\varepsilon}^{n+1}}{\bar{\varepsilon}^n}\right| \leq 1 \quad \text{for all } \phi \tag{4.2.17}$$

where $g = \bar{\varepsilon}^{n+1}/\bar{\varepsilon}^n$ is the amplification factor, and is a function of time step Δt, frequency, and the mesh size Δx. It follows from (4.2.16) that

$$g = 1 + d(e^{I\phi} - 2 + e^{-I\phi}) \tag{4.2.18a}$$

or

$$g = 1 - 2d(1 - \cos\phi) \tag{4.2.18b}$$

Thus, the stability condition is

$$g \leq 1 \tag{4.2.19}$$

or

$$1 - 2d(1 - \cos\phi) \geq -1 \tag{4.2.20}$$

Since the maximum of $1 - \cos\phi$ is 2, we arrive at, for stability,

$$0 \leq d \leq 1/2 \tag{4.2.21}$$

The von Neumann stability analysis shown above can be used to determine the computational stability properties of other finite difference schemes to be discussed subsequently.

OTHER EXPLICIT SCHEMES

Richardson Method

If the diffusion equation (4.2.1) is modeled by the form

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\alpha\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{\Delta x^2}, \quad O(\Delta t^2, \Delta x^2) \tag{4.2.22}$$

This is known as the Richardson method and is unconditionally unstable.

Dufort-Frankel Method

The finite difference equation for this method is given by

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\alpha\left(u_{i+1}^n - 2\dfrac{u_i^{n+1} + u_i^{n-1}}{2} + u_{i-1}^n\right)}{\Delta x^2} \tag{4.2.23a}$$

or

$$(1 + 2d)u_i^{n+1} = (1 - 2d)u_i^{n-1} + 2d\left(u_{i+1}^n + u_{i-1}^n\right), \quad O(\Delta t^2, \Delta x^2, (\Delta t/\Delta x)^2) \tag{4.2.23b}$$

This scheme can be shown to be unconditionally stable by the von Neumann stability analysis.

4.2.2 IMPLICIT SCHEMES

Laasonen Method

Contrary to the explicit schemes, the solution for *implicit schemes* involves the variables at more than one nodal point for the time step $(n + 1)$. For example, we may write the difference equation for (4.2.1a) in the form

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)}{\Delta x^2}, \quad O(\Delta t, \Delta x^2) \tag{4.2.24}$$

This equation is written for all grid points at $n + 1$ time step, leading to a tridiagonal form. The scheme given by (4.2.24) is known as the Laasonen method. This is unconditionally stable.

Crank-Nicolson Method

An alternative scheme of (4.2.24) is to replace the diffusion term by an average between n and $n + 1$,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\right], \quad O(\Delta t^2, \Delta x^2) \tag{4.2.25}$$

This may be rewritten as

$$A + B = C + D \tag{4.2.26}$$

where

$$A = \frac{u_i^{n+\frac{1}{2}} - u_i^n}{\Delta t/2}, \qquad B = \frac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\Delta t/2}, \qquad C = \frac{\alpha\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right)}{(\Delta x)^2},$$

$$D = \frac{\alpha\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right)}{(\Delta x)^2}$$

Note that $A = C$ and $B = D$ represent explicit and implicit scheme, respectively. This scheme is known as the Crank-Nicolson method. It is seen that $A = C$ is solved explicitly for the time step $n + 1/2$ and the result is substituted into $B = D$. The scheme is unconditionally stable.

β-Method

A general form of the finite difference equation for (4.2.1) may be written as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\left[\frac{\beta\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)}{(\Delta x)^2} + \frac{(1-\beta)\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{(\Delta x)^2}\right] \tag{4.2.27}$$

This is known as the β-method. For $1/2 \le \beta \le 1$, the method is unconditionally stable. For $\beta = 1/2$, equation (4.2.27) reduces to the Crank-Nicolson scheme, whereas $\beta = 0$ leads to the FTCS method.

A numerical example for the solution of a typical parabolic equation characterized by Couette flow is presented in Section 4.7.2.

4.2.3 ALTERNATING DIRECTION IMPLICIT (ADI) SCHEMES

Let us now examine the solution of the two-dimensional diffusion equation,

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{4.2.28}$$

with the forward difference in time and the central difference in space (FTCS). We write an explicit scheme in the form

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \alpha\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}\right), \quad O(\Delta t, \Delta x^2, \Delta y^2) \tag{4.2.29}$$

It can be shown that the system is stable if

$$d_x + d_y \le \frac{1}{2} \tag{4.2.30}$$

Here, diffusion numbers d_x and d_y are defined as

$$d_x = \frac{\alpha \Delta t}{\Delta x^2}, \qquad d_y = \frac{\alpha \Delta t}{\Delta y^2} \tag{4.2.31}$$

For simplicity, let $d_x = d_y = d$ for $\Delta x = \Delta y$. This will give $d \leq 1/4$ for stability, which is twice as restrictive. To avoid this restriction, consider an implicit scheme

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha\left(\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2}\right) \tag{4.2.32}$$

or

$$d_x u_{i+1,j}^{n+1} + d_x u_{i-1,j}^{n+1} - (2d_x + 2d_y + 1)u_{i,j}^{n+1} + d_y u_{i,j+1}^{n+1} + d_y u_{i,j-1}^{n+1} = -u_{i,j}^{n} \tag{4.2.33}$$

This leads to a pentadiagonal system.

An alternative is to use the alternating direction implicit scheme, by splitting (4.2.25) into two equations:

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\Delta t/2} = \alpha\left(\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right) \tag{4.2.34a}$$

and

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\Delta t/2} = \alpha\left(\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2}\right) \tag{4.2.34b}$$

This scheme is unconditionally stable. These two equations can be written in a tridiagonal form as follows:

$$\underbrace{-d_1 u_{i+1,j}^{n+\frac{1}{2}} + (1 + 2d_1)u_{i,j}^{n+\frac{1}{2}} - d_1 u_{i-1,j}^{n+\frac{1}{2}}}_{\text{implicit in } x\text{-direction}} = \underbrace{d_2 u_{i,j+1}^{n} + (1 - 2d_2)u_{i,j}^{n} + d_2 u_{i,j-1}^{n}}_{\text{explicit in } y\text{-direction}} \tag{4.2.35a}$$

$$\underbrace{-d_2 u_{i,j+1}^{n+1} + (1 + 2d_2)u_{i,j}^{n+1} - d_2 u_{i,j-1}^{n+1}}_{\text{unknown}} = \underbrace{d_1 u_{i+1,j}^{n+\frac{1}{2}} + (1 - 2d_1)u_{i,j}^{n+\frac{1}{2}} + d_1 u_{i-1,j}^{n+\frac{1}{2}}}_{\text{known}} \tag{4.2.35b}$$

where

$$d_1 = \frac{1}{2}d_x = \frac{1}{2}\frac{\alpha \Delta t}{\Delta x^2}$$

$$d_2 = \frac{1}{2}d_y = \frac{1}{2}\frac{\alpha \Delta t}{\Delta y^2}$$

Note that (4.2.35a) is implicit in the x-direction and explicit in the y-direction, known as the x-sweep. The solution of (4.2.35a) provides the data for (4.2.35b) so that the y-sweep can be carried out in which the solution is implicit in the y-direction and explicit in the x-direction.

4.2.4 APPROXIMATE FACTORIZATION

The ADI formulation can be shown to be an approximate factorization of the Crank-Nicolson scheme. To this end, let us write the Crank-Nicolson scheme for (4.2.25) in the form

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}=\frac{\alpha}{2}\left[\begin{array}{l}\dfrac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{\Delta x^2}+\dfrac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{\Delta x^2}\\ +\dfrac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{\Delta y^2}+\dfrac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{\Delta y^2}\end{array}\right],$$

$$O(\Delta t^2, \Delta x^2, \Delta y^2) \quad (4.2.36)$$

Introducing a compact notation,

$$\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}$$

$$\delta_y^2 u_{i,j} = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

we may rewrite (4.2.36) as

$$\left[1-\frac{1}{2}(d_x\delta_x^2+d_y\delta_y^2)\right]u_{i,j}^{n+1}=\left[1+\frac{1}{2}(d_x\delta_x^2+d_y\delta_y^2)\right]u_{i,j}^{n} \quad (4.2.37)$$

To compare (4.2.37) with the ADI formulation, we use (4.2.36) to rewrite the ADI equations as

$$\frac{u_{i,j}^{n+\frac{1}{2}}-u_{i,j}^{n}}{\frac{\Delta t}{2}}=\alpha\left(\frac{\delta_x^2 u_{i,j}^{n+\frac{1}{2}}}{\Delta x^2}+\frac{\delta_y^2 u_{i,j}^{n}}{\Delta y^2}\right) \quad (4.2.38a)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}=\alpha\left(\frac{\delta_x^2 u_{i,j}^{n+\frac{1}{2}}}{\Delta x^2}+\frac{\delta_y^2 u_{i,j}^{n+1}}{\Delta y^2}\right) \quad (4.2.38b)$$

Rearranging (4.2.38a,b)

$$\left(1-\frac{1}{2}d_x\delta_x^2\right)u_{i,j}^{n+\frac{1}{2}}=\left(1+\frac{1}{2}d_y\delta_y^2\right)u_{i,j}^{n} \quad (4.2.39a)$$

$$\left(1-\frac{1}{2}d_y\delta_y^2\right)u_{i,j}^{n+1}=\left(1+\frac{1}{2}d_x\delta_x^2\right)u_{i,j}^{n+\frac{1}{2}} \quad (4.2.39b)$$

and eliminating $u_{i,j}^{n+\frac{1}{2}}$ between (4.2.39a) and (4.2.39b),

$$\left(1-\frac{1}{2}d_x\delta_x^2\right)\left(1-\frac{1}{2}d_y\delta_y^2\right)u_{i,j}^{n+1}=\left(1+\frac{1}{2}d_x\delta_x^2\right)\left(1+\frac{1}{2}d_y\delta_y^2\right)u_{i,j}^{n} \quad (4.2.40)$$

or

$$\left[1-\frac{1}{2}(d_x\delta_x^2+d_y\delta_y^2)+\frac{1}{4}d_xd_y\delta_x^2\delta_y^2\right]u_{i,j}^{n+1}=\left[1+\frac{1}{2}(d_x\delta_x^2+d_y\delta_y^2)+\frac{1}{4}d_xd_y\delta_x^2\delta_y^2\right]u_{i,j}^{n} \quad (4.2.41)$$

We note that, compared to (4.2.37), the additional term in (4.2.41)

$$\frac{1}{4}d_xd_y\delta_x^2\delta_y^2(u_{i,j}^{n+1}-u_{i,j}^{n})$$

is smaller than the truncation error of (4.2.37). Thus, it is seen that the ADI formulation is an approximate factorization of the Crank-Nicolson scheme.

4.2.5 FRACTIONAL STEP METHODS

An approximation of multidimensional problems similar to ADI or approximate factorization schemes is also known as the method of fractional steps. This method splits the multidimensional equations into a series of one-dimensional equations and solves them sequentially. For example, consider a two-dimensional equation

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4.2.42}$$

The Crank-Nicolson scheme for (4.2.36) can be written in two steps:

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\frac{\Delta t}{2}} = \frac{\alpha}{2}\left[\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2}\right] \tag{4.2.43a}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = \frac{\alpha}{2}\left[\frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2} + \frac{u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}}{\Delta y^2}\right] + O(\Delta t^2, \Delta x^2, \Delta y^2) \tag{4.2.43b}$$

This scheme is unconditionally stable.

4.2.6 THREE DIMENSIONS

The ADI method can be extended to three-space dimensions for the time intervals $n, n+1/3, n+2/3$, and $n+1$. Consider the unsteady diffusion problem,

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{4.2.44}$$

The three-step FDM equations are written as

$$\frac{u_{i,j,k}^{n+\frac{1}{3}} - u_{i,j,k}^{n}}{\Delta t/3} = \alpha\left(\frac{\delta_x^2 u_{i,j,k}^{n+\frac{1}{3}}}{\Delta x^2} + \frac{\delta_y^2 u_{i,j,k}^{n}}{\Delta y^2} + \frac{\delta_z^2 u_{i,j,k}^{n}}{\Delta z^2}\right) \tag{4.2.45a}$$

$$\frac{u_{i,j,k}^{n+\frac{2}{3}} - u_{i,j,k}^{n+\frac{1}{3}}}{\Delta t/3} = \alpha\left(\frac{\delta_x^2 u_{i,j,k}^{n+\frac{1}{3}}}{\Delta x^2} + \frac{\delta_y^2 u_{i,j,k}^{n+\frac{2}{3}}}{\Delta y^2} + \frac{\delta_z^2 u_{i,j,k}^{n+\frac{1}{3}}}{\Delta z^2}\right) \tag{4.2.45b}$$

$$\frac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n+\frac{2}{3}}}{\Delta t/3} = \alpha\left(\frac{\delta_x^2 u_{i,j,k}^{n+\frac{2}{3}}}{\Delta x^2} + \frac{\delta_y^2 u_{i,j,k}^{n+\frac{2}{3}}}{\Delta y^2} + \frac{\delta_z^2 u_{i,j,k}^{n+1}}{\Delta z^2}\right), \quad O(\Delta t, \Delta x^2, \Delta y^2, \Delta z^2) \tag{4.2.45c}$$

This method is conditionally stable with $(d_x + d_y + d_z) \leq 3/2$. A more efficient method may be derived using the Crank-Nicolson scheme.

$$\frac{u^*_{i,j,k} - u^n_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta_x^2 u^*_{i,j,k} + \delta_x^2 u^n_{i,j,k}}{\Delta x^2} + \frac{\delta_y^2 u^n_{i,j,k}}{\Delta y^2} + \frac{\delta_z^2 u^n_{i,j,k}}{\Delta z^2}\right]$$

$$\frac{u^{**}_{i,j,k} - u^n_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta_x^2 u^*_{i,j,k} + \delta_x^2 u^n_{i,j,k}}{\Delta x^2} + \frac{1}{2}\frac{\delta_y^2 u^{**}_{i,j,k} + \delta_y^2 u^n_{i,j,k}}{\Delta y^2} + \frac{\delta_z^2 u^n_{i,j,k}}{\Delta z^2}\right]$$

$$\frac{u^{n+1}_{i,j,k} - u^n_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta_x^2 u^*_{i,j,k} + \delta_x^2 u^n_{i,j,k}}{\Delta x^2} + \frac{1}{2}\frac{\delta_y^2 u^{**}_{i,j,k} + \delta_y^2 u^n_{i,j,k}}{\Delta y^2} + \frac{1}{2}\frac{\delta_z^2 u^{n+1}_{i,j,k} + \delta_z^2 u^n_{i,j,k}}{\Delta z^2}\right] \tag{4.2.46}$$

In this scheme, the final solution $u^{n+1}_{i,j,k}$ is obtained in terms of the intermediate steps $u^*_{i,j,k}$ and $u^{**}_{i,j,k}$.

4.2.7 DIRECT METHOD WITH TRIDIAGONAL MATRIX ALGORITHM

Consider the implicit FDM discretization for the transient heat conduction equation in the form,

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}\right) \tag{4.2.47}$$

This may be rewritten as

$$a_i T_{i-1}^{n+1} + b_i T_i^{n+1} + c_i T_{i+1}^{n+1} = g_i \tag{4.2.48}$$

with

$$a_i = c_i = -\frac{\alpha \Delta t}{\Delta x^2}, \qquad b_i = 1 + \frac{2\alpha\Delta t}{\Delta x^2}, \qquad g_i = T_i^n \tag{4.2.49}$$

If Dirichlet boundary conditions are applied to this problem, we obtain the following tridiagonal form, known as tridiagonal matrix algorithm (TDMA) or Thomas algorithm [Thomas, 1949]:

$$\begin{bmatrix} b_1 & c_1 & 0 & \cdot & \cdot & \cdot & \cdot \\ a_2 & b_2 & c_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_3 & b_3 & c_3 & 0 & \cdot & \cdot \\ \cdot & \cdot & * & * & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & * & * & * & \cdot \\ \cdot & \cdot & \cdot & \cdot & * & * & c_{NI} \\ 0 & \cdot & \cdot & \cdot & \cdot & a_{NI} & b_{NI} \end{bmatrix} \begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \\ T_3^{n+1} \\ * \\ * \\ * \\ T_{NI}^{n+1} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ * \\ * \\ * \\ g_{NI} \end{bmatrix} \tag{4.2.50}$$

An upper triangular form of the tridiagonal matrix may be obtained as follows:

$$b_i = b_i - \frac{a_i}{b_{i-1}} c_{i-1} \quad i = 2, 3, \ldots NI$$

$$g_i = g_i - \frac{a_i}{b_{i-1}} g_{i-1} \quad i = 2, 3, \ldots NI$$

$$T_{NI} = \frac{g_{NI}}{b_{NI}}$$

$$T_j = \frac{g_j - c_j T_{j+1}}{b_j} \quad j = NI - 1, \quad NI - 2, \ldots, 1$$

It should be noted that Neumann boundary conditions can also be accommodated into this algorithm with the tridiagonal form still maintained.

4.3 HYPERBOLIC EQUATIONS

Hyperbolic equations, in general, represent wave propagation. They are given by either first order or second order differential equations, which may be approximated in either explicit or implicit forms of finite difference equations. Various computational schemes are examined below.

4.3.1 EXPLICIT SCHEMES AND VON NEUMANN STABILITY ANALYSIS

Euler's Forward Time and Forward Space (FTFS) Approximations

Consider the first order wave equation (Euler equation) of the form

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a > 0 \tag{4.3.1}$$

The Euler's forward time and forward space approximation of (4.3.1) is written in the FTFS scheme as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_i^n}{\Delta x} \tag{4.3.2}$$

It follows from (4.2.15) and (4.3.2) that the amplification factor assumes the form

$$g = 1 - C(e^{I\phi} - 1) = 1 - C(\cos\phi - 1) - IC\sin\phi = 1 + 2C\sin^2\frac{\phi}{2} - IC\sin\phi \tag{4.3.3}$$

with C being the Courant number or CFL number [Courant, Friedrichs, and Lewy, 1967],

$$C = \frac{a\Delta t}{\Delta x}$$

and

$$|g|^2 = g\,g^* = \left(1 + 2C\sin^2\frac{\phi}{2}\right)^2 + C^2\sin^2\phi = 1 + 4C(1 - C)\sin^2\frac{\phi}{2} \geq 1 \tag{4.3.4}$$

where g^* is the complex conjugate of g. Note that the criterion $|g| \leq 1$ for all values of ϕ can not be satisfied ($|g|$ lies outside the unit circle for all values of ϕ, Figure 4.3.1). Therefore, the explicit Euler scheme with FTFS is unconditionally unstable.